Sharp Bounds on the Spectral Radius of Nonnegative Matrices and Digraphs

Brian K. Butler[a,*], Paul H. Siegel[a]

[a]*Department of Electrical and Computer Engineering, University of California San Diego, La Jolla, CA 92093 USA*

Abstract

The spectral radius $\rho(A)$ of a matrix A is the maximum modulus of the eigenvalues. We present bounds on $\rho(A)$ that are often tighter and are applicable to a larger class of nonnegative matrices than previously reported. The bounds are particularly suited to matrices which are sparse.

We complete the paper by applying these bounds to digraphs, deriving the associated equality conditions which relate to the outdegree regularity of the digraph. Finally, we show that the equality conditions may be achieved only for very specific values of the digraph's spectral radius.

Keywords: Nonnegative matrix, Spectral radius, Bound, Digraph, Perron root

1. Introduction

Let $A = (a_{ij})$, $1 \le i, j \le n$ be an $n \times n$ nonnegative matrix. The eigenvalues of A are the complex roots of the characteristic equation $\det(A - \mu I) = 0$. The set of distinct eigenvalues is called the *spectrum* of A, denoted $\sigma(A) = \{\mu_1, \ldots, \mu_m\}$, and the *spectral radius* (or *Perron root* in this context) of A is the real number $\rho(A) = \max\{|\mu| : \mu \in \sigma(A)\}$. Recall that for a nonnegative matrix A, the spectral radius is an eigenvalue; that is, $\rho(A) \in \sigma(A)$ (see [1, p. 503]).

In this paper, we derive several new bounds on the spectral radius of nonnegative matrices which we then use to bound the spectral radius of a large class of digraphs. Our results generalize those found in Zhang & Li [2], Kolotilina [3], Xu & Xu [4], and Güngör & Das [5]. With respect to the bounds of Liu [6], we find new equality conditions when they are applied to digraphs. (For a survey of prior work on the spectral radius of digraphs, see Brualdi [7].)

The nonnegative $n \times n$ matrix A, with $n \ge 2$, is said to be *reducible* if there exists a permutation matrix P such that $PAP^T = \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix}$ where X and Z are square submatrices. Otherwise, A is said to be *irreducible*.

Let $r_i(A)$ denote the sum of the elements along the ith row of A; that is $r_i(A) = \sum_{j=1}^{n} a_{ij}$ for $i \in \{1, \ldots, n\}$. The following two classical results bound the spectral radius.

Theorem 1.1 (Frobenius). *Let $A = (a_{ij})$ be an $n \times n$ nonnegative matrix with spectral radius $\rho(A)$ and row sums $r_i(A)$, $i \in \{1, \ldots, n\}$. Then*

$$\min_i r_i(A) \le \rho(A) \le \max_i r_i(A). \tag{1}$$

Moreover, if A is an irreducible matrix, then equality holds on either side (and hence both sides) of (1) *if and only if all row sums of A are equal.*

Proof. See Minc [8, pp. 24–26]. □

Theorem 1.2 (See [1, 2]). *Let A be an $n \times n$ nonnegative matrix with spectral radius $\rho(A)$ and $x = (x_1, \ldots, x_n)^T$ be a positive column vector. Then*

$$\min_{1 \le i \le n} \left[\frac{(Ax)_i}{x_i} \right] \le \rho(A) \le \max_{1 \le i \le n} \left[\frac{(Ax)_i}{x_i} \right]. \tag{2}$$

Moreover, if A is an irreducible matrix, then equality holds on either side (and hence both sides) of (2) *if and only if the vector x is an eigenvector of A corresponding to $\rho(A)$.*

---

*Corresponding author

Email addresses: `butler@ieee.org` (Brian K. Butler), `psiegel@ucsd.edu` (Paul H. Siegel)

In the following section, we present new bounds, along with equality conditions, on the spectral radius of general nonnegative matrices. We further refine the equality conditions in Section 3 and then apply these bounds and equality conditions to general digraphs in Section 4, illustrating them with a detailed example.

2. Bounds on the Spectral Radius of Nonnegative Matrices

In this section, we characterize the spectral radius of nonnegative matrices with nonzero row sums. It is well known [4, 6] that deleting the zero rows and their corresponding columns (*i.e.*, the columns having the same indices as the zero rows) leaves unaffected the nonzero entries in the spectrum of a matrix. Since the column removal may reveal new all-zero rows, this process may have to be applied multiple times to finally produce a matrix with nonzero row sums. Once this is achieved, the bounds of this section may be applied to the reduced matrix.

Let $A = (a_{ij})$ be an $n \times n$ matrix. We denote the (i, j)th entry of matrix A^k by $a_{ij}^{(k)}$, noting that $a_{ij}^{(0)} = \delta_{ij}$, where δ_{ij} is the Kronecker delta. Let $r_i(A^k)$ denote the sum of the ith row of A^k, that is, $r_i(A^k) = \sum_{j=1}^n a_{ij}^{(k)}$. Using the fact that, for any $n \times n$ matrix B, the row sums of the product AB are given by

$$r_i(AB) = \sum_{j=1}^n \sum_{k=1}^n a_{ij} b_{jk} = \sum_{j=1}^n a_{ij} r_j(B), \tag{3}$$

one can derive additional useful row-sum expressions such as $r_i(A^k) = \sum_{j=1}^n a_{ij}^{(k-t)} r_j(A^t)$, for all $0 \le t \le k$. We will make frequent use of the column vector $x = (r_1(A^k), \dots, r_n(A^k))^T$ and the diagonal matrix $D = \mathrm{diag}(r_1(A^k), \dots, r_n(A^k))$, for some integer $k \ge 0$. Then,

$$(A^t x)_i = \sum_{j=1}^n a_{ij}^{(t)} r_j(A^k) = r_i(A^{t+k}) \tag{4}$$

and, assuming that the row sums of A^k are nonzero,

$$r_i(D^{-1} A^t D) = \frac{\sum_{j=1}^n a_{ij}^{(t)} r_j(A^k)}{r_i(A^k)} = \frac{r_i(A^{t+k})}{r_i(A^k)}, \tag{5}$$

for any $t \ge 0$ and all $i \in \{1, \dots, n\}$. Also, as Liu [6] remarked, if the row sums of a nonnegative matrix A are nonzero, then so are the row sums of A^k, for $k \ge 0$. We now prove a theorem that provides a generalization of the bounds in Xu & Xu [4]. We will need the following two lemmas, the first of which is well known.

Lemma 2.1. *Let A be an $n \times n$ nonnegative matrix with spectral radius $\rho(A)$. If A^L is irreducible, for some $L > 0$, then A is also irreducible and the positive eigenvectors of A and A^L agree up to a scale factor.*

Lemma 2.2. *Let A be an $n \times n$ matrix with spectral radius $\rho(A) \ne 0$. If, for some $k \ge 0$, $x = (r_1(A^k), \dots, r_n(A^k))^T$ is an eigenvector of A corresponding to $\rho(A)$, then so is $y = (r_1(A^{k+1}), \dots, r_n(A^{k+1}))^T$, and $y = \rho(A)x$.*

Proof. Since $(Ax)_i = \sum_{j=1}^n a_{ij} r_j(A^k) = r_i(A^{k+1}) = y_i$ for all $i \in \{1, \dots, n\}$, then $Ax = y = \rho(A)x$. $\square$

Theorem 2.3. *Let A be an $n \times n$ nonnegative matrix with spectral radius $\rho(A)$ and nonzero row sums. Then, for any integers $M > 0$, $N \ge 0$, and $k \ge 0$,*

$$\min_{\substack{1 \le i \le n \\ 1 \le j \le n}} \left[\left(\frac{r_i\left(A^{k+M}\right) r_j\left(A^{k+N}\right)}{r_i(A^k) r_j(A^k)} \right)^{\frac{1}{M+N}} : a_{ij}^{(M)} > 0 \right] \le \rho(A) \le \max_{\substack{1 \le i \le n \\ 1 \le j \le n}} \left[\left(\frac{r_i\left(A^{k+M}\right) r_j\left(A^{k+N}\right)}{r_i(A^k) r_j(A^k)} \right)^{\frac{1}{M+N}} : a_{ij}^{(M)} > 0 \right]. \tag{6}$$

Moreover, if A^{M+N} is an irreducible matrix, then equality holds in either side (and hence both sides) of (6) if and only if $x = (r_1(A^k), \dots, r_n(A^k))^T$ is an eigenvector of A.

Proof. Define the invertible diagonal matrix $D = \text{diag}(r_1(A^k), \ldots, r_n(A^k))$. Since A^{M+N} and $D^{-1}A^{M+N}D$ are similar matrices, we have

$$\rho(A^{M+N}) = \rho(D^{-1}A^{M+N}D) \leq \max_{1 \leq i \leq n} r_i\left(D^{-1}A^{M+N}D\right) \tag{7}$$

where the inequality in (7) follows from Theorem 1.1. The row sums in (7) can be formulated as

$$r_i\left(\left(D^{-1}A^M D\right)\left(D^{-1}A^N D\right)\right) = \sum_{j=1}^{n} \left(D^{-1}A^M D\right)_{ij} r_j\left(D^{-1}A^N D\right) \tag{8}$$

$$\leq r_i\left(D^{-1}A^M D\right) \max_j \left\{ r_j\left(D^{-1}A^N D\right) : a_{ij}^{(M)} > 0 \right\} \tag{9}$$

$$= \max_j \left\{ r_i\left(D^{-1}A^M D\right) r_j\left(D^{-1}A^N D\right) : a_{ij}^{(M)} > 0 \right\}, \tag{10}$$

where (8) follows from (3). The restricted maximizations in (9) and (10) make use of the fact that the *sparsity patterns* (*i.e.*, the locations of its nonzero entries) of A^M and $D^{-1}A^M D$ are the same.

Applying (5) to the factors in the product in (10), with $t = M$ and $t = N$, respectively, we conclude that

$$\rho(A)^{M+N} = \rho(A^{M+N}) \leq \max_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}} \left[\frac{r_i\left(A^{k+M}\right) r_j\left(A^{k+N}\right)}{r_i(A^k) r_j(A^k)} : a_{ij}^{(M)} > 0 \right].$$

The proof of the lower bound in (6) is completely analogous.

We now show the equality condition, assuming that A^{M+N} is irreducible. If equality holds in the upper bound of (6), then (7) must also hold with equality. Then, by the equality condition of Theorem 1.1 applied to $D^{-1}A^{M+N}D$, we conclude that, in fact, $\rho(D^{-1}A^{M+N}D) = r_i\left(D^{-1}A^{M+N}D\right)$, for all $i \in \{1, \ldots, n\}$. Referring to (5), we know that $r_i\left(D^{-1}A^{M+N}D\right) = (A^{M+N}x)_i/x_i$, for all i, where $x = (r_1(A^k), \ldots, r_n(A^k))^T$. This shows that x is a positive eigenvector of A^{M+N}. Now we apply Lemma 2.1 to show that x is an eigenvector of A, as desired.

Conversely, suppose that $x = (r_1(A^k), \ldots, r_n(A^k))^T$ is an eigenvector of A. Then Lemma 2.2 implies $\rho(A)^t = r_i(A^{k+t})/r_i(A^k)$, for all $i \in \{1, \ldots, n\}$ and $t \geq 0$. This shows that both the upper and lower bounds in (6) hold with equality. $\square$

Remark 2.1. The nonzero row-sum assumption is not required for $k = 0$.

The bounds of Xu & Xu [4] are a special case of Theorem 2.3, where $k = M = N = 1$. As another special case, in which $N = 0$, we recover the following bounds due to Liu [6].

Corollary 2.4 (Liu [3, 6]). *For any integer $L > 0$,*

$$\min_{1 \leq i \leq n} \left(\frac{r_i\left(A^{k+L}\right)}{r_i(A^k)} \right)^{1/L} \leq \rho(A) \leq \max_{1 \leq i \leq n} \left(\frac{r_i\left(A^{k+L}\right)}{r_i(A^k)} \right)^{1/L}. \tag{11}$$

Since the new bounds of Theorem 2.3 depend upon the sparsity pattern of A^M, they may produce sharper bounds than the bounds of Liu when critical entries of A are zero. However, from the proof of Theorem 2.3 it is clear that Corollary 2.4 with $L = M + N$ is at least as tight as Theorem 2.3. Nevertheless, in some applications it may be prohibitively complex to compute $r_i(A^{M+N+k})$ for Corollary 2.4 as opposed to examining the sparsity pattern of A^M and computing $r_i(A^{M+k})$ and $r_i(A^{N+k})$ as required by Theorem 2.3.

Next, we briefly generalize bounds of a similar form developed by Kolotilina [3, §5].

Theorem 2.5. *Let $A = (a_{ij})$ be an $n \times n$ nonnegative matrix with spectral radius $\rho(A)$ and row sums $r_1(A), \ldots, r_n(A)$, all nonzero. Then*

$$\min_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}} \left[\frac{r_i^\alpha(A^{k+L})r_j^{1-\alpha}(A^{k+L})}{r_i^\alpha(A^k)r_j^{1-\alpha}(A^k)} : a_{ij}^{(L)} > 0 \right] \leq \rho(A^L) \leq \max_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}} \left[\frac{r_i^\alpha(A^{k+L})r_j^{1-\alpha}(A^{k+L})}{r_i^\alpha(A^k)r_j^{1-\alpha}(A^k)} : a_{ij}^{(L)} > 0 \right] \tag{12}$$

for any integer $L \geq 1$ and α such that $0 \leq \alpha \leq 1$.

3

Proof. We apply the bounds

$$\min_{\substack{1 \le i \le n \\ 1 \le j \le n}} \left[r_i^\alpha(A) r_j^{1-\alpha}(A) : a_{ij} > 0 \right] \le \rho(A) \le \max_{\substack{1 \le i \le n \\ 1 \le j \le n}} \left[r_i^\alpha(A) r_j^{1-\alpha}(A) : a_{ij} > 0 \right],$$

which were proved in Kolotilina [3, §5], to $D^{-1} A^L D$, where the matrix $D = \text{diag}(r_1(A^k), \ldots, r_n(A^k))$. $\qquad\square$

Remark 2.2. Note that (12) with $\alpha = 0.5$ is equivalent to (6) with $L = M = N$.

Since $\rho(A^T) = \rho(A)$, the theorems and corollaries of this section and the previous may be restated in terms of the column sums and the left eigenvectors of A.

3. Further Equality Conditions on the Spectral Radius Bounds

In this section we develop alternative equality conditions for (6) by generalizing the proofs in Zhang & Li [2, §2]. Like Zhang & Li, we divide the equality conditions for these bounds into two cases corresponding to whether A^{M+N} is irreducible or reducible. We address the former first, which readily follows from (4).

Corollary 3.1 (to Theorem 2.3). *If A^{M+N} is an irreducible matrix, then equality holds on either side (and hence both sides) of (6) if and only if $\rho(A) = r_i(A^{k+1})/r_i(A^k)$ for all $i \in \{1, \ldots, n\}$.*

The case in which A^{M+N} is reducible requires some background concerning imprimitive matrices, which we review next. A nonnegative irreducible matrix A having only one eigenvalue with a modulus equal to $\rho(A)$ is said to be *primitive*. If a nonnegative irreducible matrix A has $h > 1$ eigenvalues with modulus $\rho(A)$, it is said to be *imprimitive* or a *cyclic* matrix, and h is known as the *index of imprimitivity*.

Lemma 3.2 (See [9, §3.4].). *Let A be an $n \times n$ irreducible nonnegative matrix with index of imprimitivity equal to h. Let $L > 0$ be an integer and r be the greatest common divisor (gcd) of h and L. Then A^L is reducible if and only if $r > 1$. In general there is a permutation matrix P that symmetrically permutes A^L to the block diagonal matrix*

$$PA^L P^T = \begin{pmatrix} C_1 & 0 & \cdots & 0 \\ 0 & C_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_r \end{pmatrix}, \tag{13}$$

where each C_ℓ matrix is an $n_\ell \times n_\ell$ irreducible nonnegative matrix. Furthermore, for $r > 1$, P also symmetrically permutes A to form

$$PAP^T = \begin{pmatrix} 0 & A_{12} & 0 & \cdots & 0 \\ 0 & 0 & A_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{r-1,r} \\ A_{r,1} & 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{14}$$

where the all-zero submatrices along the diagonal are square and of order $n_1, \ldots, n_r$, respectively. When (14) holds with $r > 1$, we say that A is r-cyclic. The block (i.e., submatrix) $A_{\ell,m}$ is $n_\ell \times n_m$, for all $\ell \in \{1, \ldots, r\}$ and $m = (\ell \bmod r) + 1$. Moreover,

$$C_1 = [A_{12} A_{23} \cdots A_{r-1,r} A_{r,1}]^{(L/r)}$$
$$C_2 = [A_{23} A_{34} \cdots A_{r,1} A_{12}]^{(L/r)}$$
$$\cdots$$
$$C_r = [A_{r,1} A_{12} \cdots A_{r-2,r-1} A_{r-,1r}]^{(L/r)},$$

and $\rho(A^L) = \rho(C_1) = \cdots = \rho(C_r)$.

4

Remark 3.1. Note that for a given matrix, its r value may vary depending on the specified value of L, since $r = \gcd(h, L)$.

Recall from Perron-Frobenius theory that each square nonnegative matrix A has at least one nonnegative eigenvector $x \neq 0$, such that $Ax = \rho(A)x$ [1, p. 503]. If A is irreducible, such an eigenvector x is necessarily positive and unique up to a positive scale factor. If A is reducible, it may have a positive eigenvector, or even multiple linearly independent positive eigenvectors associated with $\rho(A)$. In the case of the reducible matrix A^L of Lemma 3.2, having eigenvalue $\rho(A^L)$ with an algebraic multiplicity equal to r, there are r linearly independent positive eigenvectors of A^L corresponding to $\rho(A^L)$, a fact which we utilize to prove the next theorem.

Theorem 3.3. *Let A be an $n \times n$ irreducible nonnegative matrix with spectral radius $\rho(A)$, index of imprimitivity equal to h, and nonzero row sums. Let $M > 0$, $N \geq 0$, and $k \geq 0$ be integers and $r = \gcd(h, M + N)$. If A^{M+N} is reducible $(r > 1)$, then equality holds on either side (and hence both sides) of* (6) *if and only if*

$$\frac{r_i(A^{k+1})}{r_i(A^k)} = c_{m(i)}, \tag{15}$$

for all $i \in \{1, \ldots, n\}$, where the ith row of A has been assigned to the ℓth block, $\ell \in \{1, \ldots, r\}$, using the mapping $\ell = m(i)$ according to (13) *and* (14)*, and c_ℓ is a constant for the ℓth block. Moreover, $\rho(A)^r = \prod_{\ell=1}^{r} c_\ell$.*

Proof. Let $x = (r_1(A^k), \ldots, r_n(A^k))^T$. Without loss of generality we will assume that A is in the form of (14) and A^{M+N} is in block diagonal form (13), where each C_ℓ is $n_\ell \times n_\ell$ and irreducible. Let x be divided into r subvectors, such that $x = (w_1^T, \ldots, w_r^T)^T$ and the ℓth subvector w_ℓ has n_ℓ elements.

If either equality holds in (6), then by applying Lemma 3.2 and Theorem 1.2 block-by-block, it can be shown that x is a positive eigenvector of A^{M+N}. Now, we show that $X = \left\{ (g_1 w_1^T, \ldots, g_r w_r^T)^T : g_1, \ldots, g_r \in \mathbb{R}^+ \right\}$, is the complete set of positive eigenvectors of A^{M+N}, where $\mathbb{R}^+$ is the set of real, positive numbers. If we evaluate y in $A^{M+N}y = \rho(A^{M+N})y$, where $y \in X$, it readily reduces to $g_\ell C_\ell w_\ell = \rho(A^{M+N})g_\ell w_\ell$ or simply $C_\ell w_\ell = \rho(A^{M+N})w_\ell$, for all $\ell \in \{1, \ldots, r\}$ due to the block diagonal form of A^{M+N}. Since C_ℓ is irreducible, w_ℓ is the unique positive eigenvector of C_ℓ, up to a positive scale factor, for all $\ell \in \{1, \ldots, r\}$. Hence, there can be no other positive eigenvectors of A^{M+N} beyond set X.

Since A is irreducible, it must have a unique (up to a scale factor) positive eigenvector that we shall call y'. Additionally, y' is in the set X, because every eigenvector of A must also be an eigenvector of A^{M+N}. For the first block, with $1 \leq i \leq n_1$, we find

$$\frac{r_i(A^{k+1})}{r_i(A^k)} = \frac{(Ax)_i}{x_i} = \frac{(A_{12}w_2)_i}{(w_1)_i} = \frac{\rho(A)g_1 (w_1)_i}{g_2 (w_1)_i} = \rho(A)\frac{g_1}{g_2}, \tag{16}$$

where we take $g_1, \ldots, g_r$ to be real, positive constants that depend upon y'. Note that (16) is constant within the first block. The other blocks $(1 < \ell \leq r)$ follow similarly, confirming (15).

Conversely, suppose that (15) is true for all $i \in \{1, \ldots, n\}$. Then, for the first block $(\ell = 1)$, using (3), (4), and (15), we see that

$$\frac{r_i(A^{k+L})}{r_i(A^k)} = \frac{\sum_{j=1}^{n} a_{ij}^{(L-1)} r_j(A^{k+1})}{r_i(A^k)} = \frac{c_1 \sum_{j=1}^{n} a_{ij}^{(L-1)} r_j(A^k)}{r_i(A^k)} = \cdots = \frac{c_1 c_2 \cdots c_L \, r_i(A^k)}{r_i(A^k)} = c_1 c_2 \cdots c_L,$$

for all $1 \leq i \leq n_1$ and $1 \leq L \leq r$. For values of L greater than r the indexing of c_ℓ must wrap around to 1. Thus, for the ith row, which is in block $m(i)$, we have

$$\frac{r_i(A^{k+M})}{r_i(A^k)} = \prod_{\ell=m(i)}^{m(i)+M-1} c_{[(\ell-1) \bmod r]+1} \qquad \text{and} \qquad \frac{r_i(A^{k+N})}{r_i(A^k)} = \prod_{\ell=m(i)}^{m(i)+N-1} c_{[(\ell-1) \bmod r]+1}, \tag{17}$$

for all $i \in \{1, \ldots, n\}$. Recognize that the values of (17) are still dependent on the row index i. Next, we limit our consideration of (17) to rows i and j, respectively, such that $a_{ij}^{(M)} > 0$ as in (6). Thus, row j is in block $m(j)$, where $m(j) = [(m(i)+M-1) \bmod r]+1$. Therefore, forming the product of the row-sum ratios in (17), with such a restriction, results in row-sum ratios in the right-hand side of (6) that are independent of i and, hence, equality is true on both sides of (6) with $\rho(A)^r = \prod_{\ell=1}^{r} c_\ell$. □

4. Bounds Applied to Digraphs

In this section, we cast the results of previous sections in graph-theoretic terms and derive further equality conditions related to the integrality of the adjacency matrix. We first review some basic concepts and terminology related to digraphs. (For a more complete treatment, we refer the reader to Minc [8, §4.3] and Brualdi & Ryser [9, chap. 3].) Let $G = (V, E)$ be a directed graph, or *digraph*, with a nonempty set of n vertices, $V = \{v_1, ..., v_n\}$, and a collection E of directed edges or *arcs*. The digraph is called *simple* if it contains no self-loops or multiarcs. In contrast to the results of Zhang & Li [2], Xu & Xu [4], and Güngör & Das [5] which we generalize below, our results are not limited to simple digraphs.

The *adjacency matrix* $A(G)$ of any digraph G is the nonnegative matrix whose (i, j)th entry a_{ij} is the number of arcs directed from vertex v_i to vertex v_j in G. The *spectral radius* $\rho(G)$ of digraph G is defined to be the spectral radius of $A(G)$. In a digraph G, a *directed walk* is an alternating sequence of vertices and arcs from v_i to v_j in G such that every arc in the sequence is preceded by its initial vertex and is followed by its terminal vertex. The *length* of a directed walk is the number of arcs in the sequence. The number of distinct directed walks from v_i to v_j of length k in G is equal to the (i, j)th entry of $A(G)^k$ and is denoted by $W_k(i, j)$. The digraph G is *strongly connected* if and only if $A(G)$ is irreducible. A strongly connected digraph G is also characterized by an *index of imprimitivity* $h(G)$ which is equal to the index of imprimitivity of $A(G)$. Furthermore, a digraph G is classified as *cyclically r-partite* when $r > 1$ and r divides $h(G)$; see, for example, Brualdi & Ryser [9, §3.4].

The *outdegree* d_i^+ of vertex $v_i \in V$ in the digraph $G = (V, E)$ is defined to be the number of arcs in E with initial vertex v_i. Thus, the outdegree of vertex v_i is equal to the ith row sum of the adjacency matrix $A(G)$. This concept can be generalized to the *k-outdegree d_i^{k+}*, which is the number of directed walks of length k with starting vertex v_i. That is, $d_i^{k+} \triangleq \sum_{j=1}^n W_k(i, j)$ for any $k > 0$ and $d_i^{0+} \triangleq 1$.

A vertex v_i with no outgoing arcs (i.e., $d_i^+ = 0$) is known as a *sink*. Sinks correspond to zero rows in $A(G)$. Thus, the results of Sections 2 and 3 directly apply to general digraphs without sinks. The first theorem in this section bounds the spectral radii of these digraphs, while the final corollary will show that equality in the bounds may only be achieved for very limited values of $\rho(G)$.

We need to introduce terminology to capture the equality conditions of Section 3 in a digraph context. With respect to Corollary 3.1, we will call digraph $G = (V, E)$ *average κ-outdegree regular* if

$$\frac{d_i^{\kappa+}}{d_i^{(\kappa-1)+}} = c, \text{ for all } v_i \in V,$$

where $\kappa \geq 1$. Thus for $\kappa = 2$ our definition matches that of Zhang & Li. If G is cyclically r-partite, the set of vertices V may be partitioned into r disjoint subsets $V = V_1 \cup V_2 \cup \cdots \cup V_r$ according to (14). With respect to Theorem 3.3, we will call digraph G *average κ-outdegree r-quasiregular* if G is cyclically r-partite and

$$\frac{d_i^{\kappa+}}{d_i^{(\kappa-1)+}} = c_j, \text{ for all } v_i \in V_j$$

and all $j \in \{1, \ldots, r\}$, where $\kappa \geq 1$. To be cyclically r-partite, all arcs joining vertices in V_j either initiate in $V_{[(j-2) \bmod r]+1}$ or terminate in $V_{(j \bmod r)+1}$. Thus for $r = 2$, this condition degenerates to the bipartite-semiregular condition of Zhang & Li. For $\kappa = 1$, we suggest dropping the word "average" from these two new terms to be consistent with prior terminology.

Now we are ready to formulate the equality conditions of Theorem 2.3 in digraph terms.

Theorem 4.1. *Let $G = (V, E)$ be a digraph with spectral radius $\rho(G)$, n vertices, and no sinks. Then, for any integers $M > 0$, $N \geq 0$, and $k \geq 0$,*

$$\min_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}} \left[\left(\frac{d_i^{(k+M)+} d_j^{(k+N)+}}{d_i^{k+} d_j^{k+}} \right)^{\frac{1}{N+M}} : W_M(i, j) > 0 \right] \leq \rho(G) \leq \max_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}} \left[\left(\frac{d_i^{(k+M)+} d_j^{(k+N)+}}{d_i^{k+} d_j^{k+}} \right)^{\frac{1}{N+M}} : W_M(i, j) > 0 \right]. \quad (18)$$

Moreover, if G is strongly connected, then equality in (18) holds if and only if G is average $(k + 1)$-outdegree regular or average $(k+1)$-outdegree r-quasiregular or both, where $r = \gcd(N+M, h(G))$ and $h(G)$ is the index of imprimitivity of G.

Proof. We apply Theorem 2.3 to the adjacency matrix $A(G)$ to yield (18). Corollary 3.1 and Theorem 3.3 justify the equality conditions. □

Remark 4.1. In practice, one will likely apply (18) with $M = 1$, for simplicity in its evaluation. When $M = 1$, the condition that $W_M(i, j) > 0$ is equivalent to $\{v_i, v_j\} \in E$.

As a corollary, we find the following new equality conditions on the bounds of Liu.

Corollary 4.2. *For any integer $L > 0$ and $r = \gcd(L, h(G))$, the following bounds are satisfied with the same equality conditions as Theorem 4.1,*

$$\min_{1 \le i \le n} \left(\frac{d_i^{(k+L)+}}{d_i^{k+}} \right)^{1/L} \le \rho(G) \le \max_{1 \le i \le n} \left(\frac{d_i^{(k+L)+}}{d_i^{k+}} \right)^{1/L}. \tag{19}$$

The following lemma presents another useful result from Liu [6] that we will use shortly. It shows that, when indexed by k, the upper and lower bounds of (11) and (19) form monotonically non-increasing and non-decreasing sequences, respectively.

Lemma 4.3 (Liu [6, Theorem 3.3]). *Let A be an $n \times n$ nonnegative matrix with nonzero row sums. Then, for any integer $L > 0$,*

$$\min_{1 \le j \le n} \left[\frac{r_j\left(A^{k+L}\right)}{r_j(A^k)} \right] \le \frac{r_i\left(A^{k+1+L}\right)}{r_i(A^{k+1})} \le \max_{1 \le j \le n} \left[\frac{r_j\left(A^{k+L}\right)}{r_j(A^k)} \right],$$

for all $k \ge 0$ and $i \in \{1, \ldots, n\}$.

Lemma 4.4. *Let c be a rational number and d be a nonzero, finite real number. If the sequence $\{d \cdot c^j\}_{j=0}^{\infty}$ contains only integers, then c is also an integer.*

Proof. Let $c = p/q$, where p and $q > 0$ are coprime integers. Since $d \cdot p^j/q^j$ is an integer, d is a multiple of q^j, which cannot hold for all $j \ge 0$ unless $q = 1$. □

Theorem 4.5. *Let G be a strongly connected digraph. If equality holds in (19) for G and some $k = t$, then equality holds for all $k \ge t$ and $\rho(G)$ is the rth root of an integer, where $r = \gcd(L, h(G))$.*

Proof. Lemma 4.3 proves the first part by showing that the bounds are monotonic in k. In the case that A^L is irreducible ($r = 1$) and equality holds in (19), then by Corollary 3.1, $c = \rho(G)$ is a rational number and $d_i^{(j+t)+} = d_i^{t+}\rho(G)^j$ holds for all $j \ge 0$ and all $i \in \{1, \ldots, n\}$. Since the k-outdegree d_i^{k+} of any vertex is integral, then by Lemma 4.4, $\rho(G)$ is an integer. In the case that A^L is reducible ($r > 1$) and equality holds in (19), then by Theorem 3.3, $\rho(G)^r$ is a rational number and $d_i^{(rj+t)+} = d_i^{t+}\rho(G)^{rj}$ holds for all $j \ge 0$ and all $i \in \{1, \ldots, n\}$. Thus, $\rho(G)^r$ is an integer. □

Corollary 4.6. *Let G be a strongly connected digraph. If equality holds in (18) for G and some $M > 0$, $N \ge 0$, and $k \ge 0$, then $\rho(G)$ is the rth root of an integer, where $r = \gcd(M + N, h(G))$.*

Proof. As discussed in Section 2, the bounds of Corollary 2.4 with $L = M + N$ are at least as tight as the bounds of Theorem 2.3. Therefore, with respect to the upper bounds,

$$\rho(G)^{M+N} \le \max_{1 \le i \le n} \left[\frac{d_i^{(k+M+N)+}}{d_i^{k+}} \right] \le \max_{\substack{1 \le i \le n \\ 1 \le j \le n}} \left[\frac{d_i^{(k+M)+} d_j^{(k+N)+}}{d_i^{k+} d_j^{k+}} : W_M(i, j) > 0 \right]. \tag{20}$$

Assuming equality holds on the right side of (18), then equality holds throughout (20), and we may apply Theorem 4.5. □

Just as deleting the zero rows and their corresponding columns preserved the spectral radius in Section 2, the removal of any sinks from the digraph G leaves $\rho(G)$ undisturbed. Thus, this simple modification allows us to extend the bounds of this section to general digraphs. Additionally, removing sources from the digraph may tighten the bounds.
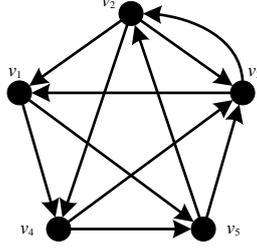
Figure 1: Digraph G_1 having adjacency matrix $A(G_1)$

Example 4.1. The order-5 example presented in [4] and [5] provides a useful illustration. Given the digraph $G_1 = (V, E)$ shown in Fig. 1, we find the 5×5 adjacency matrix to be

$$A(G_1) = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

The spectral radius of G_1 is $\rho(G_1) \approx 2.193399638$. First, we examine Corollary 4.2. Table 1 shows the quantities corresponding to each vertex $v_i \in V$ needed to evaluate (19), for all values of (k, L) such that $L + k \le 4$. The minimum and maximum of each, shown on the right side of the table, form the bounds on $\rho(G_1)$. The bounds corresponding

Table 1: Intermediate Computations and Bounds of Corollary 4.2 for all (k, L), such that $L + k \le 4$.

(k, L) **Parameters**		$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$	min	max
d_i^+	$(0,1)$	2	3	2	2	2	2	3
$\sqrt{d_i^{2+}}$	$(0,2)$	2	2.4495	2.2361	2	2.2361	2	2.4495
d_i^{2+}/d_i^{1+}	$(1,1)$	2	2	2.5	2	2.5	2	2.5
$\sqrt[3]{d_i^{3+}}$	$(0,3)$	2.0801	2.3513	2.1544	2.1544	2.2240	2.0801	2.3513
$\sqrt{d_i^{3+}/d_i^{1+}}$	$(1,2)$	2.1213	2.0817	2.2361	2.2361	2.3452	2.0817	2.3452
d_i^{3+}/d_i^{2+}	$(2,1)$	2.25	2.1667	2	2.5	2.2	2	2.5
$\sqrt[4]{d_i^{4+}}$	$(0,4)$	2.1407	2.3206	2.1657	2.1407	2.1899	2.1407†	2.3206
$\sqrt[3]{d_i^{4+}/d_i^{1+}}$	$(1,3)$	2.1898	2.1302	2.2240	2.1898	2.2572	2.1302	2.2572†
$\sqrt{d_i^{4+}/d_i^{2+}}$	$(2,2)$	2.2913	2.1985	2.0976	2.2913	2.1448	2.0976	2.2913
d_i^{4+}/d_i^{3+}	$(3,1)$	2.3333	2.2308	2.2	2.1	2.0909	2.0909	2.3333

to $(k, L) = (1, 2)$ are the tightest of the bounds here for $L + k \le 3$. When extended to $L + k = 4$, the bounds using $(k, L) = (0, 4)$ and $(1, 3)$ yield the tightest lower and upper bounds, respectively, as indicated with a "†".

Theorem 4.1, with $M = 1$, yields the bounds shown in Table 2. In three of four cases the bounds of Theorem 4.1 with $N = 1$ produced tighter bounds than Corollary 4.2 with $L = 1$. Also, the bounds indicated with a "‡" are tighter than the bounds of the first table for the same maximum order of outdegree computed.

Finally, we show the Kolotilina-based bounds of Theorem 2.5 in Table 3. Since we limited Theorem 4.1 to the case where $M = 1$ in order to keep the sparsity pattern determination simple, in evaluating bounds of Theorem 2.5 we limit consideration of L to 1. In Table 3 we have used an "*" to indicate which bounds are independent of the

Table 2: Lower and Upper Bounds on $\rho(G_1)$ from Theorem 4.1 with M=1.

(k, N)	L.B. on $\rho(G_1)$	U.B. on $\rho(G_1)$
$(0, 1)$	2	2.4495
$(0, 2)$	2	2.4662
$(1, 1)$	2	2.5
$(0, 3)$	2.0598	2.3403‡
$(1, 2)$	2.0801	2.3208‡
$(2, 1)$	2.0817	2.3717
$(0, 4)$	2.1118	2.3116
$(1, 3)$	2.1407	2.2900
$(2, 2)$	2.1204	2.2774
$(3, 1)$	2.0954	2.2815

Table 3: Lower and Upper Bounds on $\rho(G_1)$ from Theorem 2.5 with L=1.

k	L.B. on $\rho(G_1)$	U.B. on $\rho(G_1)$
0	2*	2.4495@$\alpha = 0.50$
1	2*	2.5000*
2	2.0801@$\alpha = 0.50$	2.3602@$\alpha = 0.55$
3	2.0993@$\alpha = 0.92$	2.2611@$\alpha = 0.70$

α parameter. The best lower bounds of Theorem 4.1 were equal to those produced by Corollary 4.2 but tighter than those produced by Theorem 2.5. Also, Theorem 4.1 produced tighter upper bounds compared with either Theorem 2.5 or Corollary 4.2 when the maximum order of outdegree was limited to three. However, Theorem 4.1 produced the loosest upper bounds when the maximum order of outdegree was relaxed to four.

We have generally found that the best set of parameters depends on the digraph selected. For digraphs that are sparser than G_1, the advantages of Theorems 4.1 and 2.5 will be even more evident.

We note that the bipartite condition (*i.e.*, cyclically *r*-partite with $r = 2$) was sometimes unmentioned in prior work when defining the "semiregular" digraph property [10]. Its necessity is apparent in this example. The digraph G_1 might be outdegree semiregular and average 2-outdegree semiregular by some definitions, but it is not bipartite and hence does not meet the bounds with equality.

5. Conclusions

We have generalized the bounds and equality conditions of several prior works regarding the spectral radius of nonnegative matrices and digraphs. Much of the earlier work applied to irreducible matrices and strongly-connected simple digraphs. We have generalized these to a larger set of bounds and a more general set of digraphs. Finally, we have shown that the equality conditions of the bounds, when applied to strongly connected digraphs, may only be met when the spectral radius is the *r*th root of an integer.

Acknowledgment

References

[1] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, Cambridge, 1985.

[2] X.-D. Zhang, J.-S. Li, Spectral radius of non-negative matrices and digraphs, Acta Math. Sin. 18 (2002) 293–300.

[3] L.Yu. Kolotilina, Bounds and inequalities for the Perron root of a nonnegative matrix, J. Math. Sci. 121 (2004) 2481–2507.

[4] G.-H. Xu, C.-Q. Xu, Sharp bounds for the spectral radius of digraphs, Linear Algebra Appl. 430 (2009) 1607–1612.

[5] A.D. Güngör, K.Ch. Das, Improved upper and lower bounds for the spectral radius of digraphs, Appl. Math. Comput. 216 (2010) 791–799.

[6] S.-L. Liu, Bounds for the greatest characteristic root of a nonnegative matrix, Linear Algebra Appl. 239 (1996) 151–160.

[7] R.A. Brualdi, Spectra of digraphs, Linear Algebra Appl. 432 (2010) 2181–2213.

[8] H. Minc, Nonnegative Matrices, John Wiley & Sons, New York, 1988.

[9] R.A. Brualdi, H.J. Ryser, Combinatorial Matrix Theory, Cambridge University Press, Cambridge, 1991.

[10] R.A. Brualdi, From the Editor-in-Chief: Errata to Sharp bounds for the spectral radius of digraphs, Linear Algebra Appl. 438 (2013) 3–4.