

Enhancing Binary Images of Non-Binary LDPC Codes

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Abstract—We investigate the reasons behind the superior performance of belief propagation decoding of non-binary LDPC codes over their binary images when the transmission occurs over the binary erasure channel. We show that although decoding over the binary image has lower complexity, it has worse performance owing to its larger number of stopping sets relative to the original non-binary code. We propose a method to find redundant parity-checks of the binary image that eliminate these additional stopping sets, so that we achieve performance comparable to that of the original non-binary LDPC code with lower decoding complexity.

I. INTRODUCTION

Low-density parity-check (LDPC) codes were introduced by Gallager [1] and rediscovered in the 1990's. Davey and MacKay showed that non-binary LDPC (NB-LDPC) codes perform better than binary codes for the same bit length [2]. Despite their better performance, higher decoding complexity has limited the implementation of non-binary LDPC codes in real-world applications. Reduced complexity decoding is possible using an equivalent binary Tanner graph, called the binary image of the code. There are many ways to construct binary images of non-binary codes - one way, referred to as basic binary image, is to represent each non-binary codeword by a binary vector of the same bit length. A recent result [3] defines an extended binary image of longer bit length and establishes the equivalence of belief propagation (BP) decoding over it to BP decoding over the non-binary codes for the binary erasure channel (BEC). Based on this image, we compare NB-LDPC codes to their basic binary images and suggest an algorithm to find redundant parity-checks to improve the performance of the basic binary images.

This paper is organized as follows. In Section II, we briefly describe the BP decoding of NB-LDPC codes. We define the basic and extended binary images in Section III. The superiority of BP over NB-LDPC code for BEC compared to its basic binary image is shown in Section IV and the algorithm to bridge this gap in performance through addition of redundant parity-checks to the basic binary image is given in Section V. We summarize our findings in Section VI.

II. $\text{GF}(2^b)$ -LDPC CODES

An LDPC code \mathcal{C}_q over $\text{GF}(q)$ where $q = 2^b$ is specified by a parity-check matrix \mathbf{H}_q of dimension $m_q \times n_q$ whose elements are from $\text{GF}(q)$, where the number of non-zero elements in \mathbf{H}_q is proportional to n_q . The code is then defined as the nullspace of \mathbf{H}_q , i.e.,

$$\mathcal{C}_q = \{\underline{X} \in \text{GF}(q)^{n_q} : \mathbf{H}_q \otimes_q \underline{X}^T = \underline{0}^T\}$$

where \underline{X} is assumed to be a row-vector, $\underline{0}$ the all-zero row-vector, \otimes_q denotes matrix multiplication when matrix elements belong to $\text{GF}(q)$, and T denotes transposition. When the parity-check matrix \mathbf{H}_q is full-rank, the code \mathcal{C}_q is a vector-subspace of $\text{GF}(q)^{n_q}$ of dimension $k_q = n_q - m_q$, has a block-length n_q symbols, and is therefore of rate $R = \frac{k_q}{n_q} = 1 - \frac{m_q}{n_q}$.

We associate with the matrix \mathbf{H}_q a bipartite graph, called the Tanner graph \mathcal{G}_q , as follows. Corresponding to the j^{th} column of \mathbf{H}_q is a variable node V_j^q , and corresponding to the i^{th} row is a check node C_i^q , $i \in [m_q] \triangleq \{1, 2, \dots, m_q\}$, $j \in [n_q]$. Every non-zero entry $h_{i,j}$ of \mathbf{H}_q corresponds to an edge between C_i^q and V_j^q with label $h_{i,j}$. We denote by $\mathcal{N}^v(j)$ the set of check nodes connected to a given variable node $j \in [n_q]$, i.e., the neighbors of the j^{th} variable node, and by $\mathcal{N}^c(i)$ the set of variable nodes connected to a given check node $i \in [m_q]$. When the degree of every variable node and every check node in \mathcal{G}_q is d_l and d_r respectively, i.e., $|\mathcal{N}^v(j)| = d_l$, $|\mathcal{N}^c(i)| = d_r \forall i \in [m_q], j \in [n_q]$, the LDPC code \mathcal{C}_q is said to be (d_l, d_r) -regular. It is easy to see that for a (d_l, d_r) -regular code, the rate satisfies $R = 1 - \frac{m_q}{n_q} = 1 - \frac{d_l}{d_r}$.

Belief Propagation Decoding

The messages passed around on the Tanner graphs in the BP decoder represent the *a posteriori* probabilities of the symbols of $\text{GF}(q)$. We will briefly describe the BP decoding of non-binary LDPC codes over the binary erasure channel (BEC) using the analogue of the *peeling decoder*, which for the BEC is the same as the BP decoder over \mathcal{G}_q , which we denote as BP(\mathbf{H}_q).

We assume a fixed isomorphism $\Phi_B : \text{GF}(q) \mapsto \text{GF}(2)^b$ that preserves the addition operation defined on $\text{GF}(q)$. This isomorphism is used to map codewords $\underline{X} \in \mathcal{C}_q \subset \text{GF}(q)^{n_q}$ to binary vectors $\underline{x} \in \text{GF}(2)^{bn_q}$. The vector \underline{x} is transmitted over a BEC with erasure probability ϵ and let \mathcal{E} be the index set of the erasures in the received word. At the receiver, the *a priori* set of eligible symbols $\mathcal{Q}_j^{(0)}$ for the j^{th} variable node consists of symbols which fit the received binary sequence corresponding to the j^{th} transmitted symbol. Thus $\mathcal{Q}_j^{(0)}$ has 2^a symbols if a out of the b bits in the received binary sequence corresponding to the j^{th} transmitted symbol are erased. The peeling decoder updates these sets iteratively by exchanging messages between variable and check nodes. For the BEC, the messages passed around iteratively can be represented by sets of eligible symbols. Let $\mathcal{Q}_{i,j}^{(\ell)}$ denote the set of eligible symbols sent by the i^{th} check node to the j^{th} variable node in the ℓ^{th} iteration. Then, the peeling decoder performs the following steps iteratively for $\ell \geq 1$

- *Check node processing*: For check node $C_i^q, i \in [m_q]$,

$$\mathcal{Q}_{i,j}^{(\ell)} = \sum_{j' \in \mathcal{N}^c(i) \setminus \{j\}} h_{i,j'} \mathcal{Q}_{j'}^{(\ell-1)} \quad \forall j \in \mathcal{N}^c(i).$$

- *Variable node processing*: For variable node $V_j^q, j \in [n_q]$,

$$\mathcal{Q}_j^{(\ell)} = \mathcal{Q}_j^{(\ell-1)} \cap \left(\bigcap_{i \in \mathcal{N}^v(j)} h_{i,j}^{-1} \mathcal{Q}_{i,j}^{(\ell)} \right).$$

where, if $h \in \text{GF}(q)$ and $\mathcal{Q}, \mathcal{Q}_1, \mathcal{Q}_2 \subseteq \text{GF}(q)$, we define

$$h\mathcal{Q} \triangleq \{ha \mid a \in \mathcal{Q}\}$$

$$\mathcal{Q}_1 + \mathcal{Q}_2 \triangleq \{a_1 + a_2 \mid a_1 \in \mathcal{Q}_1, a_2 \in \mathcal{Q}_2\}.$$

The decoder stops when $|\mathcal{Q}_j^{(\ell)}| = |\mathcal{Q}_j^{(\ell-1)}| \forall j \in [n_q]$. Decoding is successful if $|\mathcal{Q}_j^{(\ell)}| = 1 \forall j \in [n_q]$ for some $\ell \in \mathbb{N}$. It should be noted that any set of eligible symbols ($\mathcal{Q}_j^{(\ell)}$ or $\mathcal{Q}_{i,j}^{(\ell)}$) is a coset of a vector subspace of $\text{GF}(q)$ [4].

III. BINARY IMAGES

Since codes defined over $\text{GF}(2^b)$ can be written as a collection of vectors over $\text{GF}(2)$, we can consider the *binary images* of these codes. Further, since BP decoding for $\text{GF}(q)$ has a high computational complexity, we can make use of the binary images to decode the non-binary code.

A. Basic Binary Image

The isomorphism Φ_B defined in Section II maps any row-vector of symbols $\underline{X} = (X_1, \dots, X_{n_q}) \in \text{GF}(q)^{n_q}$ to a binary row-vector $\Phi_B(\underline{X}) \triangleq (\Phi_B(X_1), \dots, \Phi_B(X_{n_q}))$ of length bn_q . The *basic binary image* of \mathcal{C}_q is defined as the set $\mathcal{C}_B \subset \text{GF}(2)^{bn_q}$ where $\mathcal{C}_B = \{\Phi_B(\underline{X}) : \underline{X} \in \mathcal{C}_q\}$. As described earlier, the codewords of \mathcal{C}_B are the ones that are transmitted over the BEC. Note that there are $\prod_{l=0}^{b-1} (2^b - 2^l)$ possible choices for the isomorphism Φ_B .

Once fixed, the isomorphism Φ_B identifies a mapping $\Psi_m : \text{GF}(q) \mapsto \mathcal{M}_b$, where \mathcal{M}_b is the collection of all invertible matrices over $\text{GF}(2)$ of size $b \times b$, defined below. For any $h \in \text{GF}(q)$,

$$\Psi_m(h) = \begin{pmatrix} \Phi_B(h \times \Phi_B^{-1}(\underline{u}_{1,b})) \\ \Phi_B(h \times \Phi_B^{-1}(\underline{u}_{2,b})) \\ \vdots \\ \Phi_B(h \times \Phi_B^{-1}(\underline{u}_{b,b})) \end{pmatrix}^T$$

where $\underline{u}_{i,b}$ denotes the binary row-vector of length b with the element i as 1 and other elements equal to 0, i.e., the *unit vector* along the i^{th} dimension. The set $\{\Psi_m(h) : h \in \text{GF}(q)\} \subset \mathcal{M}_b$ constitutes a field under matrix addition and matrix multiplication operations over $\text{GF}(2)$, i.e., for $h_1, h_2 \in \text{GF}(q)$, $\Psi_m(h_1 + h_2) = \Psi_m(h_1) \oplus \Psi_m(h_2)$ and $\Psi_m(h_1 h_2) = \Psi_m(h_1) \otimes \Psi_m(h_2)$. Also, it is easy to see that for any $h, x \in \text{GF}(q)$, $\Psi_m(h) \otimes \Phi_B(x)^T = \Phi_B(hx)^T$.

The mappings Φ_B and Ψ_m allow us to identify \mathcal{C}_B as a linear block code with parity-check matrix \mathbf{H}_B that is obtained by replacing each element of \mathbf{H}_q by its image given by Ψ_m . We could perform BP on the Tanner graph of \mathbf{H}_B , denoted

by $\text{BP}(\mathbf{H}_B)$, instead of the more complex $\text{BP}(\mathbf{H}_q)$. However this has worse performance in general due to the large number of short cycles introduced by $\Psi_m(h)$'s in the parity-check matrix \mathbf{H}_B . In particular, for the BEC, we shall show later that this bad performance can be attributed to a larger number of stopping sets of \mathbf{H}_B .

B. Extended Binary Image

The observation made at the end of the previous subsection leads us to the question whether it is possible to design a binary image of a non-binary code that matches the performance of the non-binary code. This question was answered in the affirmative by Savin in [3] for the BEC. We briefly describe the construction of this code, called the *extended binary image*.

We define a mapping $\Phi_E : \text{GF}(q) \mapsto \text{GF}(2)^{2^b-1}$ such that each symbol of $\text{GF}(q)$ is mapped to a codeword of the *simplex code* $\mathcal{C}_S(b)$ of length $2^b - 1$, i.e., the dual of the Hamming code of blocklength $2^b - 1$. For $X \in \text{GF}(q)$,

$$\Phi_E(X) \triangleq \Phi_B(X) \otimes_2 \mathbf{G}_S(b)$$

where $\mathbf{G}_S(b)$ is the generator matrix of the simplex code of size $b \times (2^b - 1)$. By linearity of Φ_B , $\Phi_E(X_1 + X_2) = \Phi_E(X_1) \oplus \Phi_E(X_2)$. Note that the columns of $\mathbf{G}_S(b)$ are all vectors of weight $1 \leq w \leq b$. We will let the i^{th} column of $\mathbf{G}_S(b)$ be $(g_{1,i}, g_{2,i}, \dots, g_{b,i})^T$ where $i = \sum_{j=1}^b g_{j,i} 2^{j-1}$ for $i \in [2^b - 1]$. We use the mapping Φ_E to map row-vector of symbols $\underline{X} = (X_1, \dots, X_{n_q}) \in \text{GF}(q)^{n_q}$ to a binary row-vector $\Phi_E(\underline{X}) \triangleq (\Phi_E(X_1), \dots, \Phi_E(X_{n_q}))$ of length $(2^b - 1)n_q$. The ordering of the columns of $\mathbf{G}_S(b)$ implies that for $i \in [n_q], j \in [b]$

$$(\Phi_B(X_i))_j = (\Phi_E(X_i))_{2^{j-1}}, \quad (1)$$

where we write $(\Phi_B(X))_e$ to denote the e^{th} element of the vector representing $\Phi_B(X)$ for $X \in \text{GF}(q)$. The *extended binary image* of \mathcal{C}_q is defined as the set $\mathcal{C}_E \subset \text{GF}(2)^{(2^b-1)n_q}$ where $\mathcal{C}_E = \{\Phi_E(\underline{X}) : \underline{X} \in \mathcal{C}_q\}$. For a fixed Φ_E , one can define a mapping $\Psi_M : \text{GF}(q) \mapsto \mathcal{M}_{2^b-1}$ such that

$$\Psi_M(h) \otimes_2 \Phi_E(X)^T = \Phi_E(hX)^T \quad \forall h, X \in \text{GF}(q).$$

Then the following can be shown.

Lemma 1: $\Psi_M(h)$ is a permutation matrix for all $h \in \text{GF}(q)$.

The matrix \mathbf{H}_{PM} of dimension $m_q(2^b - 1) \times n_q(2^b - 1)$ is defined as one obtained by replacing each element of \mathbf{H}_q by its image under Ψ_M . Then, the Tanner graph of \mathbf{H}_{PM} is a *graph cover* [5] of \mathcal{G}_q with all edge labels equal to 1. It is easy to show that the extended binary image is

$$\mathcal{C}_E = \{\underline{x} \in \text{GF}(2)^{(2^b-1)n_q} : \mathbf{H}_{PM} \otimes_2 \underline{x}^T = \underline{0}^T\} \cap \mathcal{C}_S(b)^{n_q}$$

where

$$\mathcal{C}_S(b)^{n_q} = \underbrace{\mathcal{C}_S(b) \times \dots \times \mathcal{C}_S(b)}_{n_q} \subset \text{GF}(2)^{(2^b-1)n_q}.$$

The overall parity-check matrix for the extended binary code is therefore of the form

$$\mathbf{H}_E = \begin{pmatrix} \mathbf{H}_{PM} \\ \mathbf{I}_{n_q} \odot \mathbf{H}_S(b) \end{pmatrix}$$

where \mathbf{I}_{n_q} is the $(n_q \times n_q)$ identity matrix, $\mathbf{H}_S(b)$ is the parity-check matrix of the simplex code and \odot represents the *Kronecker product*.

As a consequence of Equation (1), transmitting the codewords of \mathcal{C}_B amounts to transmitting bits indexed by the set $\mathcal{I}_t = \{i \in [(2^b - 1)n_q] : i = 2^j \bmod (2^b - 1), j \in \{0, 1, \dots, b-1\}\}$ of the corresponding codewords in \mathcal{C}_E and puncturing the rest. We will let $\mathcal{I}_p = [(2^b - 1)n_q] \setminus \mathcal{I}_t$ denote the index set of punctured bits in \mathcal{C}_E . For ease of notation, we will define $x_{i,j}^B \triangleq (\Phi_B(X_i))_j$ and $x_{i,j}^E \triangleq (\Phi_E(X_i))_j$.

Example 2: Let us consider a code over $\text{GF}(8)$ given by the following parity-check matrix $\mathbf{H}_q = (\alpha \quad \alpha^2 \quad 1)$, where α is a *primitive element* of $\text{GF}(8)$ with the *primitive polynomial* being $x^3 + x + 1 = 0$. The code may be represented as solutions to the following linear constraint $\alpha X_1 + \alpha^2 X_2 + X_3 = 0$. Table I

$X \in \text{GF}(8)$	$\Phi_B(X)$
0	(0, 0, 0)
α	(0, 0, 1)
α^3	(0, 1, 0)
α^6	(1, 0, 0)
α^2	(1, 1, 1)
α^4	(1, 1, 0)
α^5	(1, 0, 1)
α^7	(0, 1, 1)

gives the chosen mapping Φ_B . Then the parity equations for the basic binary code can be written as

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_{1,1}^B \\ x_{1,2}^B \\ x_{1,3}^B \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_{2,1}^B \\ x_{2,2}^B \\ x_{2,3}^B \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_{3,1}^B \\ x_{3,2}^B \\ x_{3,3}^B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \text{ i.e.,}$$

$$(x_{1,2}^B + x_{1,3}^B) + x_{2,2}^B + x_{3,1}^B = 0, \quad (2)$$

$$(x_{1,1}^B + x_{1,2}^B + x_{1,3}^B) + x_{2,3}^B + x_{3,2}^B = 0, \quad (3)$$

$$(x_{1,1}^B + x_{1,3}^B) + (x_{2,1}^B + x_{2,2}^B) + x_{3,3}^B = 0. \quad (4)$$

The parity-check equation for the extended binary image can be written as

$$\Psi_M(\alpha)\Phi_E(X_1) + \Psi_M(\alpha^2)\Phi_E(X_2) + \Psi_M(1)\Phi_E(X_3) = 0.$$

Thus, the seven binary parity-check equations represented by the equation above are represented by \mathbf{H}_{PM} given by

j	$(\Phi_E(X_1))_j$							$(\Phi_E(X_2))_j$							$(\Phi_E(X_3))_j$						
	1	2	3	4	5	6	7	1	2	3	4	5	6	7	1	2	3	4	5	6	7
P_1						1							1								1
P_2							1					1									1
P_3	1												1								1
P_4					1							1									1
P_5				1				1													1
P_6		1												1							1
P_7						1							1								1

Here, equation P_1 is $x_{1,6}^E + x_{2,2}^E + x_{3,1}^E = 0$ which together with the simplex constraints $x_{1,6}^E = x_{1,2}^B + x_{1,3}^B$, $x_{2,2}^E = x_{2,2}^B$ and $x_{3,1}^E = x_{3,1}^B$ can be written as Equation (2). Similarly, equations P_2 and P_4 correspond to Equations (3) and (4) respectively. Equations P_3, P_5, P_6, P_7 are all linear combinations of the above equations.

In [3], it was shown that the extended binary image was equivalent to the non-binary code from which it was constructed. But this equivalence was established for the BEC with BP updates at the parity-check nodes and ML updates at the simplex nodes. In order to achieve this equivalence with BP, we will assume that the parity-check matrices representing the simplex codes have redundant parity-checks assuring ML performance with BP. For the simplex codes, this is achieved by forming the parity-check matrix with rows of weight 3 [6]. Hence, $\mathbf{H}_S(b)$ is the parity-check matrix of size $(2^b - 1)(2^{b-1} - 1) \times (2^b - 1)$ for the simplex code. We will denote BP over the Tanner graph of \mathbf{H}_E with $\mathbf{H}_S(b)$ having redundant checks and when only bits indexed by \mathcal{I}_t are transmitted as $\text{BP}(\mathbf{H}_E)$.

IV. SUPERIORITY OF $\text{BP}(\mathbf{H}_q)$ OVER $\text{BP}(\mathbf{H}_B)$

From the construction of the basic binary image and the extended binary image, it is clear that for each parity-check equation in \mathbf{H}_q , there are b parity-check equations in \mathbf{H}_B and $(2^b - 1)$ parity-check equations in the \mathbf{H}_{PM} portion of \mathbf{H}_E .

Lemma 3: For each row l_q in the parity-check matrix \mathbf{H}_q , the parity-check matrix \mathbf{H}_E contains all non-trivial linear combinations of the b rows corresponding to l_q in the parity-check matrix \mathbf{H}_B .

Discussion: First, note that the extra variables in the extended binary image were defined such that all non-trivial linear combinations of the transmitted bits were represented by a variable and these linear relations were maintained through the use of the simplex code for each non-binary symbol. Second, observe that by using the permutation matrices, we obtained a set of $(2^b - 1)$ parity-check equations each involving exactly one of the variables from each simplex codeword. Putting these two together, we can show the above result.

Since it is known that the performance of $\text{BP}(\mathbf{H}_q)$ is the same as that of $\text{BP}(\mathbf{H}_E)$, it suffices to show that $\text{BP}(\mathbf{H}_E)$ has superior performance in comparison with $\text{BP}(\mathbf{H}_B)$. The performance of codes over the BEC with BP decoding is determined by the *stopping sets* of the Tanner graph of their parity-check matrix [7]. By an *erasure pattern* \mathcal{E} , we mean the index set of the erasures in a vector. For a binary parity-check

matrix \mathbf{H} , let $\beta(\mathbf{H}, \mathcal{E})$ denote the erasure pattern obtained at the end of decoding the received word with erasure pattern \mathcal{E} using BP on the Tanner graph of \mathbf{H} . Then a stopping set is an erasure pattern \mathcal{E} such that $\beta(\mathbf{H}, \mathcal{E}) = \mathcal{E}$. Let $\mathbb{S}(\mathbf{H}) = \{\mathcal{E} : \beta(\mathbf{H}, \mathcal{E}) = \mathcal{E}\}$ be the set of stopping sets of \mathbf{H} . Notice that this condition is the same as the graph-theoretic requirement that every check node neighbor of variables in the Tanner graph of \mathbf{H} indexed by \mathcal{E} be connected to these variables at least twice. However, with the extended binary image, we are interested in erasure patterns only among the transmitted variables. Hence we do not have a corresponding graph-theoretic definition of stopping sets of $\text{BP}(\mathbf{H}_E)$.

Definition 4 (Stopping Sets of $\text{BP}(\mathbf{H}_E)$): Consider BP decoding of the extended binary code over the BEC with the parity-check matrix \mathbf{H}_E . A stopping set, \mathcal{E} , is a subset of the index set of transmitted bits such that BP decoding can recover no transmitted bit in \mathcal{E} , i.e.,

$$\mathcal{E} \subset \mathcal{I}_t \text{ such that } \beta(\mathbf{H}_E, \mathcal{E} \cup \mathcal{I}_p) \cap \mathcal{I}_t = \mathcal{E}.$$

Discussion: Let us denote by \mathbb{S}_E the set of all stopping sets of the extended binary image as defined above. Let $\mathbb{S}(\mathbf{H}_E)$ denote the set of all stopping sets of the code with parity-check matrix \mathbf{H}_E assuming all bits are transmitted. Define

$$\mathbb{S}^t(\mathbf{H}_E) = \{\mathcal{S} \cap \mathcal{I}_t : \mathcal{S} \in \mathbb{S}(\mathbf{H}_E)\}.$$

Then, $\mathbb{S}_E = \mathbb{S}^t(\mathbf{H}_E)$. This is because for each $\mathcal{S} \in \mathbb{S}^t(\mathbf{H}_E)$, $\beta(\mathbf{H}_E, \mathcal{S})$ is the maximal stopping set $\mathcal{S}' \in \mathbb{S}(\mathbf{H}_E)$ such that $\mathcal{S}' \cap \mathcal{I}_t = \mathcal{S}$. Note that for the basic binary image, the set of all stopping sets is just the normal definition $\mathbb{S}_B = \mathbb{S}(\mathbf{H}_B)$ since every bit of the basic binary image is transmitted.

Proposition 5 (BP(\mathbf{H}_E) better than BP(\mathbf{H}_B): $\mathbb{S}_E \subset \mathbb{S}_B$.

Proof: Suppose to the contrary that $\exists \mathcal{S} \in \mathbb{S}_E \setminus \mathbb{S}_B$. Then, there is a parity-check equation in \mathbf{H}_B that involves only one erasure, say $x_{i,e}^B = (\Phi_B(X_i))_e$. Let this equation be

$$\begin{aligned} \sum_{j \in \mathbb{I}_i} (\Phi_B(X_i))_j + \sum_{k \in \mathbb{I}} \sum_{j \in \mathbb{I}_k} (\Phi_B(X_k))_j &= 0 \\ \Rightarrow \sum_{j \in \mathbb{I}_i} x_{i,j}^B + \sum_{k \in \mathbb{I}} \sum_{j \in \mathbb{I}_k} x_{k,j}^B &= 0 \end{aligned} \quad (5)$$

for some index sets $\mathbb{I}_i, \mathbb{I}_k \subset [b]$, $\forall k \in \mathbb{I}, \mathbb{I} \subset [n_q] \setminus \{i\}$ and $e \in \mathbb{I}_i$. We know from the construction of the extended binary image that there exist variables

$$\begin{aligned} x_{k,\Sigma_k}^E &\triangleq (\Phi_E(X_k))_{\sum_{j \in \mathbb{I}_k} 2^{j-1}} = \sum_{j \in \mathbb{I}_k} (\Phi_B(X_k))_j \\ &= \sum_{j \in \mathbb{I}_k} (\Phi_E(X_k))_{2^{j-1}}, k \in \mathbb{I} \cup \{i\}. \end{aligned}$$

From Lemma 3, we also know that the equation

$$x_{i,\Sigma_i}^E + \sum_{k \in \mathbb{I}} x_{k,\Sigma_k}^E = 0 \quad (6)$$

is contained in \mathbf{H}_{PM} . Since every bit except $x_{i,e}^B$ was known in Equation (5), every bit except x_{i,Σ_i}^E is known in Equation

(6). Hence the extended binary image can solve for x_{i,Σ_i}^E . But the simplex portion of \mathbf{H}_E contains the equation

$$x_{i,\Sigma_i}^E = \sum_{j \in \mathbb{I}_i} x_{i,2^{j-1}}^E = \sum_{j \in \mathbb{I}_i} (\Phi_B(X_i))_j$$

which has only one unknown $x_{i,2^{e-1}}^E = x_{i,e}^B$, which can also be solved for. This is a contradiction to the assumption that $\mathcal{S} \in \mathbb{S}_E$. Therefore, $\mathbb{S}_E \setminus \mathbb{S}_B = \emptyset$. ■

Example 6: Consider the code in Example 2 and let the received word be (?0? 0?0 000), where ? denotes an erasure. It is easy to see that this received word corresponds to a stopping set of the basic binary image. However, the extended binary image can recover all erasures. Using P_6 we can recover $x_{2,7}^E$, then using the simplex equation $x_{2,7}^E = x_{2,1}^E + x_{2,2}^E + x_{2,4}^E$ we obtain $x_{2,2}^E$. In turn, $x_{2,i}^E, \forall i \in \{1, \dots, 7\}$ can be obtained. Finally we can use P_3 and P_7 to recover $x_{1,1}^E$ and $x_{1,4}^E$.

Note that although we have only shown improper containment of \mathbb{S}_E in the set of stopping sets of \mathbf{H}_B , in most cases this containment is proper, i.e., $\mathbb{S}_E \subsetneq \mathbb{S}_B$. One case where $\mathbb{S}_E = \mathbb{S}_B$ is when the Tanner graph corresponding to \mathbf{H}_B is cycle-free.

Proposition 5 gives us an analytical insight into why non-binary codes perform well. However, since the superiority is established only in comparison with the basic binary image corresponding to the non-binary code, the result constitutes only a partial answer to why non-binary codes are better than their binary counterparts in general.

V. ENHANCING \mathcal{C}_B USING REDUNDANT PARITY-CHECKS

It is known that the performance of a linear code over the BEC with BP can be improved by using a parity-check matrix with redundant parity-checks (RPCs). In fact, by adding enough parity-checks, we can guarantee ML performance with BP (See [6] and references therein). A similar notion is that of *stopping redundancy* [8], where RPCs are added to remove stopping sets of size smaller than the minimum distance of the code. In the same spirit, we pose the question whether it is possible to achieve the performance of a non-binary code by its basic binary image with some RPCs.

Proposition 7: For every $\mathcal{S} \in \mathbb{S}_B \setminus \mathbb{S}_E$, there exists an RPC for the basic binary image, $\underline{h}_{\mathcal{S}} \otimes_2 \Phi_B(\underline{X})^T = \underline{0}$, such that $\mathcal{S} \notin \mathbb{S}(\hat{\mathbf{H}}_B)$ where $\hat{\mathbf{H}}_B^T = (\mathbf{H}_B^T | \underline{h}_{\mathcal{S}}^T)$.

Proof: Since $\mathcal{S} \notin \mathbb{S}_E$, the BP decoder working over \mathbf{H}_E can solve for some erased transmitted bit $x_{i,2^{e-1}}^E = x_{i,e}^B$. This implies that the ML decoder working with \mathbf{H}_B can also solve for $x_{i,e}^B$. Since ML decoding over the BEC is the same as Gaussian elimination, the above means that there exists a linear combination of the parity-check equations in \mathbf{H}_B which has only $x_{i,e}^B$ as the unknown, which can be set as $\underline{h}_{\mathcal{S}}$. ■

Note that since there might be multiple linear combinations of parity-check equations of \mathbf{H}_B that can solve for $x_{i,e}^B$, the RPC $\underline{h}_{\mathcal{S}}$ is not always unique.

We now describe an algorithm that finds the RPC $\underline{h}_{\mathcal{S}}$, pseudocode for which is given as Algorithm 1. Given \mathcal{S} , the algorithm starts the peeling decoding over \mathbf{H}_E with the erasure pattern $\mathcal{S} \cup \mathcal{I}_p$. The decoder maintains a list of all the punctured

Algorithm 1 Algorithm to find \underline{h}_S given $S \in \mathbb{S}_B \setminus \mathbb{S}_E$

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1:  $\mathcal{R} \leftarrow \emptyset, \mathcal{L} \leftarrow \emptyset, \underline{h}_S \leftarrow \underline{0}$ 
2: while  $\exists$  a row  $\underline{h}_j^E \in \mathbf{H}_E$  that solves  $(\Phi_E(\underline{X}))_i = \mathbf{0}$ 
3:    $\mathcal{R} \leftarrow \mathcal{R} \cup \{(i, j)\}$ 
4:   if  $i \in \mathcal{I}_t$  then
5:      $\mathcal{L} \leftarrow \{(i, j)\}$ 
6:     while  $\mathcal{L} \neq \emptyset$  do
7:        $(i, j) \leftarrow \text{pop}(\mathcal{L})$ 
8:       Let  $\mathbb{I}_j \subset [(2^b - 1)n_q] : \sum_{k \in \mathbb{I}_j} \underline{u}_{k, (2^b - 1)n_q} = \underline{h}_j^E$ 
9:       for all  $i' \in \mathbb{I}_j \cap \mathcal{I}_p \setminus \{i\}$  do
10:        Locate  $(i', j') \in \mathcal{R}$ 
11:        push( $\mathcal{L}, (i', j')$ )
12:       end for
13:       for all  $k \in \mathbb{I}_j \cap \mathcal{I}_t$  do
14:         $\sigma_k \leftarrow \lfloor \frac{k}{2^b - 1} \rfloor b + (k \bmod (2^b - 1))$ 
15:         $\underline{h}_S \leftarrow \underline{h}_S \oplus_2 \underline{u}_{\sigma_k, bn_q}$ 
16:       end for
17:     end while
18:   return  $\underline{h}_S$ 
19:   terminate
20: end if
21: end while

```

bits that were solved and the unerased transmitted bits that were used to solve them. When a transmitted but erased bit is solved, the decoder is terminated and the algorithm finds an equation involving only this recovered transmitted bit and some unerased transmitted bits chosen based on the punctured bits that appear in the computation tree for this recovered transmitted bit. Thus, for a given stopping set, the algorithm uses a partial peeling decoding attempt over the extended binary image and a traversal through the computation tree of a recoverable transmitted bit to obtain the corresponding RPC of \mathbf{H}_B . For a given collection of stopping sets, this algorithm is first run for low-weight stopping sets since RPCs that eliminate low-weight stopping sets may also eliminate higher weight stopping sets containing those low-weight stopping sets as a subset. It is possible to optimize the choice of RPCs to minimize the degrees of the additional check nodes introduced by them. However, this optimization was not considered in the implementation used for this paper. Note that the idea here is different from the one in [9] where the authors try to optimize the set of transmitted bits under the assumption that bits of the extended binary image indexed by a subset of \mathcal{I}_p are also transmitted to obtain a lower rate code that performs better.

Tables II and III list the number of codewords and number of stopping sets for two non-binary LDPC codes, \mathcal{C}_4 and \mathcal{C}_8 , and their binary images. Tanner graph for \mathcal{C}_4 is constructed randomly with parameters $q = 2^2, n_q = 96, k_q = 32$, and $d_l = 2, d_r = 3$. Tanner graph for \mathcal{C}_8 is constructed using Progressive Edge-Growth (PEG) [10] with parameters $q = 2^3, n_q = 100, k_q = 50$, and $d_l = 2, d_r \simeq 4$. Let A_B^w denote the number of codewords of weight w in the basic binary image \mathcal{C}_B . Let $\mathbb{S}^w(\mathbf{H})$ denote the set of stopping sets of \mathbf{H} of weight

w , \mathbb{S}_E^w the set of stopping sets of the extended binary image of weight w . For \mathcal{C}_4 , RPCs were added to its basic binary image to remove extra stopping sets of weights up to 12 in \mathbf{H}_B^{12} and weights up to 15 in \mathbf{H}_B^{15} while, for \mathcal{C}_8 , RPCs were added to remove extra stopping sets of weights up to 8 in \mathbf{H}_B^8 and weights up to 10 in \mathbf{H}_B^{10} . The stopping sets $\mathbb{S}_B = \mathbb{S}(\mathbf{H}_B)$ were found using the algorithm in [11] and whether these were in \mathbb{S}_E was verified by running the BP decoder over \mathbf{H}_E . For those stopping sets in $\mathbb{S}_B \setminus \mathbb{S}_E$, the RPCs were found using the algorithm described earlier in this section. For \mathcal{C}_4 , the basic binary image has 192 variable nodes and 128 check nodes, while the number of RPCs in \mathbf{H}_B^{12} and \mathbf{H}_B^{15} are 93 and 154, respectively. For comparison, the extended binary image has 288 variable nodes and 288 check nodes. Similarly, the basic binary image for \mathcal{C}_8 has 300 variable nodes and 150 check nodes, while the number of RPCs in \mathbf{H}_B^8 and \mathbf{H}_B^{10} is 60 and 185, respectively. The number of variable and check nodes in the extended binary image for \mathcal{C}_8 is 700 and 1050, respectively. In general, for a NB-LDPC code over $\text{GF}(q)$, the number of variable and check nodes in the basic binary image is $O(\log q)$ while it is $O(q)$ in the extended binary image. The number of variable nodes in the enhanced basic binary image is the same as basic binary image, while the number of check nodes is the sum of the number of check nodes in the basic binary image and the additional RPCs added.

TABLE II
STOPPING SET WEIGHTS FOR A $\text{GF}(4)$ CODE, \mathcal{C}_4 , AND ITS BINARY IMAGES

w	A_B^w	$ \mathbb{S}_E^w $	$ \mathbb{S}^w(\mathbf{H}_B) $	$ \mathbb{S}^w(\mathbf{H}_B^{12}) $	$ \mathbb{S}^w(\mathbf{H}_B^{15}) $
≤ 4	0	0	0	0	0
5	4	4	6	4	4
6	2	2	11	2	2
7	7	7	31	7	7
8	7	9	50	9	9
9	3	4	86	4	4
10	12	23	171	23	23
11	22	30	343	30	30
12	63	80	873	80	80
13	87	120	2199	151	120
14	122	204	5463	287	204
15	205	418	13891	650	418
16	369	793	35209	1322	821

TABLE III
STOPPING SET WEIGHTS FOR A $\text{GF}(8)$ CODE, \mathcal{C}_8 , AND ITS BINARY IMAGES

w	A_B^w	$ \mathbb{S}_E^w $	$ \mathbb{S}^w(\mathbf{H}_B) $	$ \mathbb{S}^w(\mathbf{H}_B^8) $	$ \mathbb{S}^w(\mathbf{H}_B^{10}) $
≤ 2	0	0	0	0	0
3	0	0	9	0	0
4	0	0	0	0	0
5	0	0	0	0	0
6	1	1	59	1	1
7	5	5	82	5	5
8	13	13	509	13	13
9	38	38	2781	156	38
10	64	64	11763	612	64
11	143	147	45310	2102	310
12	309	358	169120	6363	871
13	799	970	663617	19346	2575
14	1906	2525	2727519	62032	7893

For the codes under consideration, Figures 1 and 2 plot the BP performance of the non-binary codes, their basic binary

images, and the enhanced binary images. The improvement in performance with RPCs is evident in the figures. The FERs of the non-binary code and the enhanced binary image are close to each other, and the corresponding BERs are also comparable.

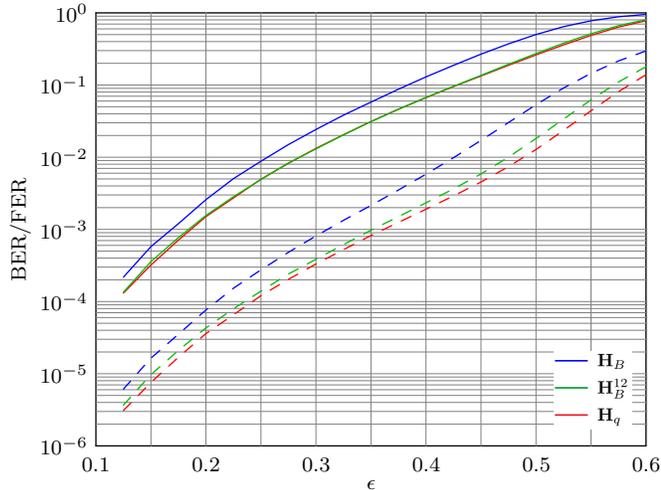


Fig. 1. Bit (dashed curves) and frame (solid curves) error rates for the random non-binary code \mathcal{C}_4 with parameters $q = 2^2, n_q = 96, k_q = 32, d_l = 2, d_r = 3$ and its binary images considered in Table II.

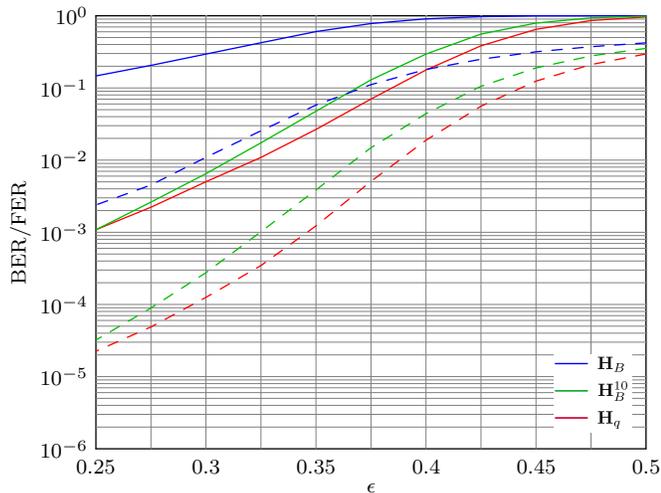


Fig. 2. Bit (dashed curves) and frame (solid curves) error rates for the PEG constructed non-binary code \mathcal{C}_8 with parameters $q = 2^3, n_q = 100, k_q = 50, d_l = 2, d_r \simeq 4$ and its binary images considered in Table III.

VI. CONCLUSION

We showed that when the transmission occurs over the BEC, BP decoding over the non-binary graph has a better performance than BP decoding over the Tanner graph of the basic binary image of the code. We proposed an algorithm to efficiently find effective redundant parity-checks for the basic binary image by observation of the BP decoding iterations of the extended binary image. Through numerical results and simulations, the effectiveness of the proposed algorithm was established. Obtaining bounds on the number of RPCs of the basic binary image needed to achieve the same performance as BP over non-binary code would be of interest. Similar characterization of the reasons for the superiority of non-binary codes over other channels involving errors as well as erasures will give further insight on designing strong codes and efficient decoding algorithms for such channels.

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