On Joint Modulation and Coding for Intersymbol Interference Channels

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Overview

- Information Rates of binary-input, ISI channels.
- Power of output sequences, and the Low-Rate Shannon Limit.
- Concatenations of linear (coset) codes w/ shaping codes.
  - Achievable Rates.
  - Biphase code example.
  - Greedy-search for simple codes.
Information Rates of ISI Channels

- **ISI channel model:**

\[
y_k = \sum_{i=0}^{\nu} h_i x_{k-i} + n_k,
\]

- \( x_k \in \{\pm 1\}, \text{ and } n_k \sim \mathcal{N}(0, \sigma^2) \) and i.i.d.

- **Mutual information rate:**

\[
I(X, Y) = \lim_{n \to \infty} \frac{1}{n} I(X_1^n; Y_1^n)
\]

- Capacity = \( \max I(X, Y) \) over all input distributions.

- **Calculating Information Rates:**

  - *For a finite-state input process, \( I(X, Y) \) can be efficiently estimated using the BCJR algorithm* [Pfister, et al., and Arnold and Loeliger, 2001].
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Information Rates of ISI Channels (con’t)

Dicode Channel, \( h(D) = \frac{1}{\sqrt{2}} (1 - D) \).

![SNR vs Achievable Rate Graph]

Achievable w/ Shaping

Achievable w/ i.i.d. inputs

SNR Per Information Bit, \( E_b/N_0 \) (dB)
Q: How much power can we get out of the channel with binary-inputs?

- One can represent outputs from the ISI filter with a state graph.
- **Definitions:**
  - The *power* of a length-$N$ output sequence, $y$, is
    \[ G(y) = \frac{1}{N} \|y\|^2. \]
  - A *cycle* is a path which begins and ends at the same state.
  - A *simple cycle* is a cycle with unique edges.
  - $G_{opt}$ is the maximum output power over all simple cycles.
- **Observation:** No sequence can have power greater than $G_{opt}$ as length $N \to \infty$. 
Example for $h(D) = 1 - D$.

- simple cycle $(-1, 1, -1)$
- inputs $(1, -1)$
- outputs $(2, -2)$
- $G_{opt} = 4$
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**Implications to the Low Rate Shannon Limit**

- The **Low-Rate Shannon Limit** (LRSL) is the minimum $E_b/N_0$, or SNR per information bit, required for reliable communication at any nonzero rate.
  - It is well-known that the LRSL is $\ln 2 = -1.59$ dB on AWGN channel.

- Since no sequence of codes (with $N \to \infty$) can realize a power gain greater than $G_{opt}$, the minimum required $E_b/N_0$ on binary-input ISI channels is **at least**

  $$\ln 2/G_{opt} = -1.59 - 10\log_{10}(G_{opt}) \text{ dB}$$

  i.e., *ISI filter provides a gain (or loss) of $G_{opt}$.*

- Comparison with channel peak, $G_{max} = \max_\omega |h(e^{j\omega})|^2$:

<table>
<thead>
<tr>
<th>Channel</th>
<th>$h(D)$</th>
<th>$G_{max}$</th>
<th>$G_{opt}$</th>
<th>Gap (dB)</th>
<th>Input Cycle</th>
</tr>
</thead>
<tbody>
<tr>
<td>PR4</td>
<td>$(1 - D^2)/\sqrt{2}$</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>-1,-1,1,1,1</td>
</tr>
<tr>
<td>EPR4</td>
<td>$(1 - D)(1 + D)^2/2$</td>
<td>64/27</td>
<td>2</td>
<td>0.75</td>
<td>-1,-1,1,1,1</td>
</tr>
<tr>
<td>E²PR4</td>
<td>$(1 - D)(1 + D)^3/\sqrt{10}$</td>
<td>27/10</td>
<td>12/5</td>
<td>0.51</td>
<td>-1,-1,-1,1,1,1</td>
</tr>
</tbody>
</table>

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Optimal simple cycles can be used to achieve \( \ln 2 / G_{opt} \).

- Let the \( x_{opt} (y_{opt}) \) be the inputs (outputs) of an optimal simple cycle.

- **Modulation:**
  - For \( u \in \{ \pm 1 \} \), map \( u \rightarrow x_u = u(p, x_{opt}) \).
    - Sequence \( p \), of length-\( \nu \), puts channel in first state of optimal cycle.
    - If cycle length is \( L \), rate penalty is \( 1 / (L + \nu) \).

- **Demodulation:**
  - Discard the header \( p \): \( r = uy_{opt} + n \)
  - System is equivalent to a BIAWGN channel with LRSL

\[
\frac{L + \nu}{L} \ln 2 / G_{opt}
\]

\((L \text{ can be made arbitrarily large by repeating the cycle}).\)
Consider a finite-state machine (FSM) modulation code $S$ which maps data bits $\{U_k\}$ to channel input bits $\{X_k\}$:

- Let $\{U_k\}$ be \textit{i.i.d. and Bernoulli one-half}. This then induces a distribution on $\{X_k\}$ for which we estimate $I(X; \mathcal{Y})$. The overall rate of the channel with shaping code is

$$R < R(S) \cdot I(X; \mathcal{Y})$$

- It is achievable with coding on bits $\{U_k\}$ using:
  - an outer linear coset code $C$ and maximum-likelihood decoding.
  - multilevel-codes/multistage-decoding.
  - \textit{For FSM codes, the BCJR-algorithm can be used to estimate} $I(X; \mathcal{Y})$. 


Dicode channel with a biphase code, i.e., \( u \rightarrow x = (u, -u) \).
Joint Modulation and Coding Example (con’t)

- No “shaping gain” realized until $R < 0.4$. Interestingly, this helps explain result from Souvignier\(^1\), comparing two serial-turbo schemes on PR4 at $R = 8/18 = 0.444$:  
  1. Inner and outer codes are convolutional codes.  
  2. Inner code is the biphase code, outer is convolutional.  

  \textit{Scheme 1 outperforms 2 by almost 3 dB at } P_b = 10^{-5} \text{.}

- Realizing almost all of the “shaping gain” requires no turbo-equalization; i.e., MLC/MSD with $m = 1$ represents the system:

\begin{center}
\begin{tikzpicture}[node distance=2cm, auto]
  \node (code) at (0,0) {Code};
  \node (biphase) [right of=code] {Biphase};
  \node (dicode) [right of=biphase] {Dicode Channel};
  \node (app) [right of=dicode] {APP};
  \node (siso) [right of=app] {Code SISO};
  \draw [->] (code) -- (biphase);
  \draw [->] (biphase) -- (dicode) node [midway, above] {\textit{(Dicode and Biphase)}};
  \draw [->] (dicode) -- (app);
  \draw [->] (app) -- (siso);
\end{tikzpicture}
\end{center}

Simple Shaping Code Design

• **Greedy-search** for rate $p : q$ shaping codes with *high average output power*.

1. Let $G$ be a graph which represents channel output sequences. Form $G^q$.

2. For each state $s$ in $G^q$:
   - Prune all but $2^p$ outgoing edges with the highest power.
   - Relabel inputs on these remaining edges with $0, \ldots, 2^p - 1$. 

Results of greedy-search.

![Graph showing achievable rate vs. SNR per information bit, $E_b/N_0$ (dB)].
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Summary

- **Characterized the Low-Rate Shannon Limit for binary-input ISI channels.**
  - For some channels, threshold equals that of real-inputs.

- **Examined the concatenation of a linear (coset) code and a FSM modulation code.**
  - Determined an achievable-rate region.
  - Designed simple shaping codes by pruning higher-power channel graph.

- **Possible Future Work:**
  - Design encoders which map into the “typical set” of an optimized Markov chain.
  - Incorporate with multilevel-coding/multistage-decoding, and optimize constituent low-density parity-check codes.
Supplement: Calculating Information Rates

- Numerical method\(^2\) for estimating \(I(\mathcal{X}, \mathcal{Y})\):

1. Sample entropies converge (Shannon-McMillan-Breiman):

   \[
   - \frac{1}{n} \log_2 p(y_1^n) + \frac{1}{n} \log_2 p(y_1^n|x_1^n) \to I(\mathcal{X}, \mathcal{Y}) \quad a.s.
   \]

2. \(p(y_i|y_1^{i-1})\) = normalization constant in forward-recursion of BCJR.
   Use to get sample entropy,

   \[
   - \frac{1}{n} \log_2 p(y_1^n) = - \frac{1}{n} \sum_{i=1}^{n} \log_2 p(y_i|y_1^{i-1}) .
   \]

- Can be extended to any finite-state machine (FSM) input process.

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